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The Vervaat Process in L_p Spaces

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It is well known that, asymptotically, the appropriately normalized Vervaat process behaves like one half times the squared empirical process. Considering these two processes as elements of the L_p-space, $1 \leqslant p < \infty$, we give a complete description of the strong and weak asymptotic behaviour of the L_p-distance between them, and thus of the L_p-norm of the Vervaat process as well. The herein obtained results also raise a number of further mathematical and probabilistic problems which we formulate as suggestions for future consideration. © 2001 Academic Press

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1. INTRODUCTION AND PRELIMINARY RESULTS

For the sake of better understanding the nature of the problems we will deal with in Sections 2 and 3, we introduce notations and discuss here some old and new results that are related to, and also provide better understanding of, our main results in Section 2.

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Let $U_1, U_2, ...$ be independent copies of a random variable U uniformly distributed over the interval $[0, 1]$. Let

$$E_n(t) := \frac{1}{n} \sum_{k=1}^{n} \mathbf{I}\{U_k \leqslant t\}, \qquad 0 \leqslant t \leqslant 1,$$

denote the empirical distribution function based on $U_1, U_2, ..., U_n$, where $\mathbf{I}$ denotes the indicator function, and let E_n^{-1} be the left-continuous inverse of E_n. We denote the empirical and quantile processes over the interval $[0, 1]$ by the following formulas

$$\beta_n^U := E_n - I,$$
$$\gamma_n^U := E_n^{-1} - I,$$

respectively, where I is the identity function $I(t) = t$. The sum

$$R_n^U := \beta_n^U + \gamma_n^U$$

of the empirical and quantile processes is known in the literature as the Bahadur–Kiefer process (cf. Bahadur, 1966, Kiefer, 1970).

Remark 1. Traditionally, it is the normalized versions $\sqrt{n}\,\beta_n^U$, $\sqrt{n}\,\gamma_n^U$ and $\sqrt{n}\,R_n^U$ that are called, respectively, the $[(0, 1)$-uniform$]$ empirical, quantile and Bahadur–Kiefer processes. We do not follow this tradition in the current paper for the sake of avoiding re-normalizations of normalized processes, and thus normalizing all the processes in this paper only once, when needed.

Vervaat (1972b) proved the following result.

THEOREM 1.1 (Vervaat, 1972b). *The statement*

$$a_n R_n^U \to_d Y, \qquad n \to \infty$$

(*with* "$\to_d$" *denoting convergence in distribution*) *cannot hold true in the space* $D[0, 1]$ (*endowed with the Skorohod topology*) *for any sequence* $\{a_n\}$ *of positive real numbers and any non-degenerate random element* Y *of* $D[0, 1]$.

Vervaat's (1972b) proof of Theorem 1.1 was based, in a most crucial and elegant way, on the following integrated Bahadur–Kiefer process

$$V_n^U(t) := \int_0^t R_n^U(s)\, ds, \qquad 0 \leqslant t \leqslant 1,$$

that has since then become known as the $[(0, 1)$-uniform$]$ Vervaat process.

We are now to recall Vervaat's (1972a,b) geometrical interpretation of $V_n^U(t)$. Namely, let

$$\Gamma_n := \{(x, y) \in [0, 1] \times [0, 1] : E_n(x-0) \leqslant y \leqslant E_n(x+0)\}$$

denote the graph of the empirical distribution function E_n. We clearly see (cf. Fig. 1) that $V_n^U(t)$ is the volume of area B between the graph Γ_n and the horizontal and vertical lines going through the points t of the corresponding coordinate axes.

The just given geometrical interpretation of $V_n^U(t)$ suggests that, when $n \to \infty$, the random variable $V_n^U(t)$ asymptotically behaves like a half of the volume of the rectangle with the four corner points $(t, E_n(t))$, (t, t), $(E_n^{-1}(t), t)$, $(E_n^{-1}(t), E_n(t))$. That is to say, the Vervaat process V_n^U asymptotically behaves like $-\frac{1}{2}\beta_n^U \gamma_n^U$. This observation, in turn, implies that V_n^U asymptotically behaves like a half of the squared empirical process, since the quantile process γ_n^U asymptotically behaves like $-\beta_n^U$. A rigorous mathematical formulation of these (somewhat implicitly found in Vervaat, 1972a,b) statements now follows.

THEOREM 1.2 (Vervaat, 1972a,b). *The two statements*

$$\frac{n}{\log\log n} \left\| V_n^U - \frac{1}{2}\{\beta_n^U\}^2 \right\|_\infty \to_{a.s.} 0, \tag{1.1}$$

$$n \left\| V_n^U - \frac{1}{2}\{\beta_n^U\}^2 \right\|_\infty \to_P 0 \tag{1.2}$$

hold true when $n \to \infty$, where $\|\cdot\|_\infty$ denotes the sup-norm, $\to_{a.s.}$ stands for "almost sure," and $\to_P$ for "in probability" convergence.

Statement (1.1) immediately implies the following fundamental corollary concerning the strong asymptotic behaviour of the Vervaat process V_n^U.

COROLLARY 1.1 (Vervaat, 1972a,b). *The set*

$$\left\{\frac{n}{\log\log n} V_n^U, n \in \mathbf{N}\right\} \tag{1.3}$$

is relatively compact in $C[O, 1]$ equipped with the sup-norm $\|\cdot\|_\infty$. Furthermore, the set of all limit points of sequence (1.3) coincides, almost surely, with the set $\{f^2 : f \in \mathscr{F}\}$, where $\mathscr{F}$ is the Finkelstein class of all absolutely continuous functions f on $[0, 1]$ such that $f(0) = 0 = f(1)$ and $\|f'\|_2 \leqslant 1$, where $\|\cdot\|_2$ denotes the L_2-norm.

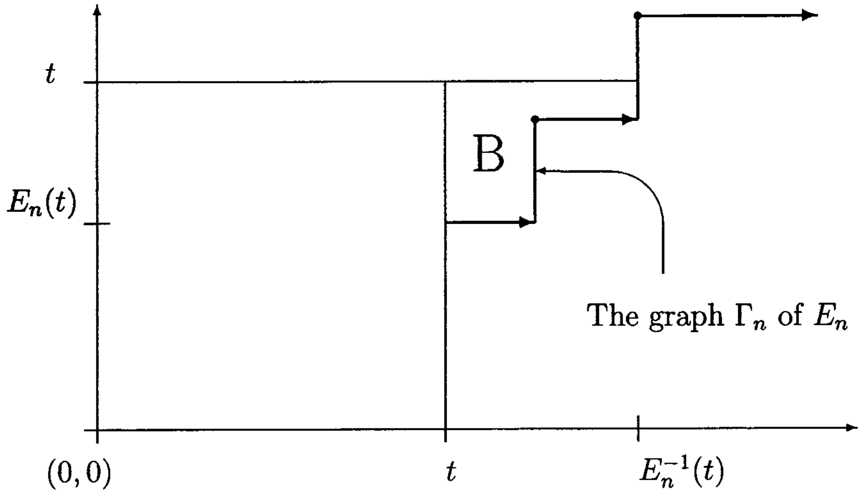

FIG. 1. The volume of area B is $V_n^U(t)$.

Corollary 1.1, in turn, implies a number of interesting results that we are now to discuss. Finkelstein (1971) proved that

$$\sup_{f \in \mathscr{F}} \|f\|_2 = \frac{1}{\pi}. \tag{1.4}$$

In view of (1.4), Corollary 1.1 immediately implies the following result

$$\limsup_{n \to \infty} \frac{n}{\log \log n} \|V_n^U\|_1 = \frac{1}{\pi^2} = 0.101321... \quad a.s., \tag{1.5}$$

where $\|\cdot\|_1$ denotes the L_1-norm. This result can be extended to the general case of L_p-norms, $p \geqslant 1$, as well. Indeed, Csörgö, Shi and Yor [CsShY] (1998) proved that

$$\sup_{f \in \mathscr{F}} \|f\|_r = c_3(r) := \frac{r^{1/2}(r+2)^{(r-2)/(2r)}}{2^{(r-1)/r} B(1/2, 1/r)} \tag{1.6}$$

for any $r \in [1, \infty)$, where $\|\cdot\|_r$, denotes the L_r-norm, and B denotes the Beta function. In view of (1.6), Corollary 1.1 implies that, for any $p \in [1, \infty)$,

$$\limsup_{n \to \infty} \frac{n}{\log \log n} \|V_n^U\|_p = \{c_3(2p)\}^2 \quad a.s. \tag{1.7}$$

Obviously, in the case $p=1$, statement (1.7) is identical to (1.5), as it should be. Furthermore, we obtain using statement (1.7) with $p=4/3$ that

$$\limsup_{n\to\infty} \frac{n}{\log\log n} \|V_n^U\|_{4/3} = \frac{2^{5/2}7^{1/4}\{\Gamma(3/4)\}^2}{3^{5/4}\{\Gamma(3/8)\}^4} = 0.110841\ldots \quad a.s. \tag{1.8}$$

We note in passing that the reason for considering the case $p=4/3$ in this paper will only become clear in the next section (cf. the paragraph just above statement (2.17) and the paragraph below statement (2.19)). For the time being, we may consider (1.8) as a mere curiosity. In the Hilbertian $p=2$ case, statement (1.7) is of natural interest and reads as follows

$$\limsup_{n\to\infty} \frac{n}{\log\log n} \|V_n^U\|_2 = \frac{2^2 3^{1/2}\pi}{\{\Gamma(1/4)\}^4} = 0.125963\ldots \quad a.s. \tag{1.9}$$

Even though the case $p=\infty$ or, in other words, the case of the sup-norm $\|\cdot\|_\infty$ is not covered by statement (1.7), using the result of Finkelstein (1971) saying that

$$\sup_{f\in\mathscr{F}} \|f\|_\infty = \tfrac{1}{2}, \tag{1.10}$$

we immediately get from Corollary 1.1 that

$$\limsup_{n\to\infty} \frac{n}{\log\log n} \|V_n^U\|_\infty = \frac{1}{4} \quad a.s. \tag{1.11}$$

As to the weak asymptotic behaviour of the Vervaat process V_n^U, statement (1.2) of Theorem 1.2 immediately implies the following corollary.

COROLLARY 1.2 (Vervaat, 1972a,b). *We have*

$$nV_n^U \to_d \tfrac{1}{2}\mathscr{B}^2 \tag{1.12}$$

in the space $C[O,1]$, *where* $\mathscr{B}$ *denotes a Brownian bridge on* $[0,1]$.

We are now to derive from Corollary 1.2 several results of independent interest. We first recall that according to Smirnov (1937) (cf. also Anderson and Darling, 1952) we have the equality

$$\begin{aligned}&\mathbf{P}\{\|\mathscr{B}\|_2 \leqslant x\}\\ &\quad = 1 + \frac{2}{\pi}\sum_{k=1}^{\infty}(-1)^k \int_{(2k-1)\pi}^{2k\pi} \frac{1}{\sqrt{-t\sin t}} \exp\left\{-\frac{1}{2}t^2x^2\right\} dt, \qquad x>0.\end{aligned} \tag{1.13}$$

Obviously, (1.13) and Corollary 1.2 imply the following result

$$\lim_{n\to\infty} \mathbf{P}\{n\,\|V_n^U\|_1 \leqslant x\} = 1 + \frac{2}{\pi}\sum_{k=1}^{\infty}(-1)^k \int_{(2k-1)\pi}^{2k\pi} \frac{1}{\sqrt{-t\sin t}} \exp\{-t^2 x\}\,dt, \qquad x>0. \tag{1.14}$$

Furthermore, Kolmogorov (1933) (cf. also Smirnov, 1939) proved that

$$\mathbf{P}\{\|\mathscr{B}\|_\infty \leqslant x\} = 1 + 2\sum_{k=1}^{\infty}(-1)^k \exp\{-2k^2x^2\}, \qquad x>0. \tag{1.15}$$

Consequently, in view of (1.15), Corollary 1.2 implies that

$$\lim_{n\to\infty} \mathbf{P}\{n\,\|V_n^U\|_\infty \leqslant x\} = 1 + 2\sum_{k=1}^{\infty}(-1)^k \exp\{-4k^2x\}, \qquad x>0. \tag{1.16}$$

To conclude, we note that while we have

$$\lim_{n\to\infty} \mathbf{P}\{n\,\|V_n^U\|_p \leqslant x\} = \mathbf{P}\{\|\mathscr{B}\|_{2p} \leqslant \sqrt{2x}\} \tag{1.17}$$

by Corollary 1.2, there are no explicit formulas of this result available yet for any $p \in (1, \infty)$. Hence, we do not know an explicit formula in the Hilbertian $p=2$ case either.

2. MAIN RESULTS

The Bahadur–Kiefer process $\gamma_n^U + \beta_n^U$ can be interpreted as the remainder term R_n^U in the representation

$$\gamma_n^U = -\beta_n^U + R_n^U \tag{2.1}$$

of the quantile process γ_n^U in terms of the empirical process β_n^U. It is well known that the remainder term R_n^U, i.e. the Bahadur–Kiefer process, is asymptotically smaller than the main term β_n^U, i.e. the empirical process, in both the L_p- and sup-topologies. We refer, for example, to Csörgő and Szyszkowicz (1998), and Csörgő and Shi (1998) for complete accounts on these developments.

In a similar vein, we can think about the process

$$Q_n^U := V_n^U - \tfrac{1}{2}\{\beta_n^U\}^2$$

(that appears in both statements (1.1) and (1.2) of Theorem 1.2) as the remainder term Q_n^U in the following representation

$$V_n^U = \tfrac{1}{2}\{\beta_n^U\}^2 + Q_n^U \tag{2.2}$$

of the Vervaat process V_n^U in terms of one half times the squared empirical process. It is well-known (cf. Zitikis, 1998, for details and references) that the remainder term Q_n^U in (2.2) is asymptotically smaller than the main term $\frac{1}{2}\{\beta_n^U\}^2$. Thus, just like in the case of R_n^U, one may like to know how small the remainder term Q_n^U is.

For the sake of discussing our main results (cf., for example, Theorem 2.3 below), we quote here the two major ones concerning the Bahadur–Kiefer process R_n^U that inspired our present investigations.

The first result gives the best possible strong, i.e., almost sure (a.s.), convergence result for the sup-norm $\|\cdot\|_\infty$ of the Bahadur–Kiefer process R_n^U (cf. Theorem 1.A and the comments that follow it in Kiefer, 1970).

THEOREM 2.1 (Kiefer, 1970). *We have*

$$\lim_{n\to\infty} \frac{n^{1/2}}{(\log n)^{1/2}} \frac{\|R_n^U\|_\infty}{\|\beta_n^U\|_\infty^{1/2}} \left[= \lim_{n\to\infty} \frac{n^{1/4}}{(\log n)^{1/2}} \frac{\|n^{1/2}R_n^U\|_\infty}{\|n^{1/2}\beta_n^U\|_\infty^{1/2}} \right] = 1 \quad a.s. \tag{2.3}$$

In particular, statement (2.3) *implies the law of the iterated logarithm*

$$\limsup_{n\to\infty} \frac{n^{3/4}}{(\log n)^{1/2}(\log\log n)^{1/4}} \|R_n^U\|_\infty = \frac{1}{2^{1/4}} \quad a.s., \tag{2.4}$$

the "other" law of the iterated logarithm

$$\liminf_{n\to\infty} \frac{n^{3/4}(\log\log n)^{1/4}}{(\log n)^{1/2}} \|R_n^U\|_\infty = \frac{\pi^{1/2}}{8^{1/4}} \quad a.s., \tag{2.5}$$

and convergence in distribution

$$\lim_{n\to\infty} \mathbf{P}\left\{ \frac{n^{3/4}}{(\log n)^{1/2}} \|R_n^U\|_\infty \leqslant x \right\} = \mathbf{P}\{\|\mathscr{B}\|_\infty^{1/2} \leqslant x\}, \qquad x>0, \tag{2.6}$$

of the sup-norm $\|\cdot\|_\infty$ *of the Bahadur–Kiefer process* R_n^U.

The part of statement (2.3) which is given in brackets is the traditional form of stating this result. Similar adjustments for statements (2.4)–(2.6) are also immediate. For a summary of further related developments we refer, for example, to Csörgő and Szyszkowicz (1998).

In order to formulate the next result, that is due to Csörgő and Shi (1998), we need some additional notation. First, we set

$$c_0(p) := \sqrt{2}\left\{\frac{1}{\sqrt{\pi}}\,\Gamma\left(\frac{p+1}{2}\right)\right\}^{1/p} \tag{2.7}$$

with Γ standing for the Gamma function. We note in passing that $c_0(p) = (\mathbf{E}\,|G|^p)^{1/p}$ where G stands for a Gaussian random variable with mean 0 and variance 1. Second, we set

$$c_4(r) := \inf_{X \in \mathscr{C}} \{\mathbf{E}\,|X|^r\}^{1/r},$$

where $\mathscr{C}$ denotes the class of all random variables having absolutely continuous distribution functions with densities f such that

$$\frac{1}{8}\int_{\mathbf{R}} \frac{(f'(x))^2}{f(x)}\,dx \leqslant 1.$$

According to Csörgő and Shi (1998), the explicit value of the constant $c_4(r)$ is not yet known in general, except for the following two cases $r = 1$ and 2 when we have

$$c_4(1) = \frac{2^{1/2}\,|a_1'|^{3/2}}{3^{3/2}},$$

with $a_1' < 0$ denoting the largest real root of the derivative $Ai'(\cdot)$ of the Airy function $Ai(\cdot)$, and

$$c_4(2) = \tfrac{1}{2^{3/2}}.$$

We are now in the position to state the best possible strong convergence result, as well as its most important consequences, for the L_p-norm $\|\cdot\|_p$ of the Bahadur–Kiefer process R_n^U.

THEOREM 2.2 (Csörgő and Shi, 1998). *For any* $p \in [2, \infty)$, *we have*

$$\lim_{n\to\infty} n^{1/2}\frac{\|R_n^U\|_p}{\|\beta_n^U\|_q^{1/2}} \left[= \lim_{n\to\infty} n^{1/4}\frac{\|n^{1/2}R_n^U\|_p}{\|n^{1/2}\beta_n^U\|_q^{1/2}}\right] = c_0(p) \quad a.s., \tag{2.8}$$

where $q := p/2$. *In particular, statement* (2.8) *implies the law of the iterated logarithm*

$$\limsup_{n\to\infty} \frac{n^{3/4}}{(\log\log n)^{1/4}}\|R_n^U\|_p \left[= \limsup_{n\to\infty} \frac{n^{1/4}}{(\log\log n)^{1/4}}\|n^{1/2}R_n^U\|_p\right]$$
$$= 2^{1/4}c_0(p)\sqrt{c_3(q)} \quad a.s., \tag{2.9}$$

the "other" law of the iterated logarithm

$$\liminf_{n\to\infty} n^{3/4}(\log\log n)^{1/4}\ \|R_n^U\|_p\left[=\liminf_{n\to\infty} n^{1/4}(\log\log n)^{1/4}\ \|n^{1/2}R_n^U\|_p\right]$$
$$=c_0(p)\sqrt{c_4(q)}\quad a.s., \tag{2.10}$$

and convergence in distribution

$$\lim_{n\to\infty}\mathbf{P}\{n^{3/4}\ \|R_n^U\|_p\leqslant x\}\left[=\lim_{n\to\infty}\mathbf{P}\{n^{1/4}\ \|n^{1/2}R_n^U\|_p\leqslant x\}\right]$$
$$=\mathbf{P}\{c_0(p)\ \|\mathscr{B}\|_q^{1/2}\leqslant x\},\qquad x>0, \tag{2.11}$$

of the L_p-norm, $\|\cdot\|_p$ of the Bahadur–Kiefer process R_n^U.

The bracketed parts of statements (2.8)–(2.11) are the very forms used by Csörgő and Shi (1998).

If we compare Theorems 2.1 and 2.2, we immediately notice substantial differences in the asymptotic behaviour of $\|R_n^U\|_\infty$ and $\|R_n^U\|_p$. First of all, the rate of strong convergence of $\|R_n^U\|_\infty$ is slower by $(\log n)^{1/2}$ than that of $\|R_n^U\|_p$ for any $p\geqslant 2$. Furthermore, in the sup-case we compare $\|R_n^U\|_\infty$ with $\|\beta_n^U\|_\infty^{1/2}$ (cf. statement (2.3)), whereas in the L_p-case we compare $\|R_n^U\|_p$ with $\|\beta_n^U\|_q^{1/2}$ (cf. statement (2.8)), where $q=p/2$. Finally, the limiting constants in statements (2.3) and (2.8) are also different. The just described differences, in turn, result in different laws of the iterated logarithm, "other" laws of the iterated logarithm, and convergence in distribution for the sup- and L_p-norms of the Bahadur–Kiefer process R_n^U (cf. statements (2.4)–(2.6) with (2.9)–(2.11), respectively). In this regard it is worth noting that the respective rates of (2.9, (2.10) and (2.11) are $(\log n)^{1/2}$ times faster than those of (2.4), (2.5) and (2.6). Roughly speaking, this is due to the use of L_p norms that make the erratic behaviour of the Bahadur–Kiefer process, R_n^U, $(\log n)^{1/2}$ times "less visible" than when using sup-norms.

Based on this discussion, one naturally suspects that there should also be substantial differences between the asymptotic behaviours of the sup- and L_p-norms of the process Q_n^U. For a preliminary investigation in this regard we refer to Remark 2.1 below towards the end of this section.

The following theorem is the best possible strong, i.e., almost sure, convergence result for $\|Q_n^U\|_p$.

THEOREM 2.3. *For any $p\in[1,\infty)$, we have*

$$\lim_{n\to\infty} n^{1/2}\frac{\|Q_n^U\|_p}{\|\beta_n^U\|_q^{3/2}}\left[=n^{1/4}\frac{\|nQ_n^U\|_p}{\|n^{1/2}\beta_n^U\|_q^{3/2}}\right]=\frac{1}{\sqrt{3}}c_0(p)\quad a.s., \tag{2.12}$$

where $q:=3p/2$.

It is remarkable and quite unexpected that the rates of convergence in both statements (2.8) and (2.12) are identical. The crucial difference between the two statements, however, is that in (2.8) we compare $\|R_n^U\|_p$ to $\|\beta_n^U\|_q^{1/2}$, $q=p/2$, whereas in (2.12) we compare $\|Q_n^U\|_p$ to $\|\beta_n^U\|_q^{3/2}$, $q=3p/2$. Of course, the limiting constants in (2.8) and (2.12) are also different. Furthermore, while Theorem 2.2 holds true for only $p\geqslant 2$, Theorem 2.3 holds true for $p\geqslant 1$. All these differences inevitably result in different laws of the iterated logarithm, "other" laws of the iterated logarithm, and convergence in distribution (cf. Corollaries 2.1–2.3, respectively) of $\|R_n^U\|_p$ and $\|Q_n^U\|_p$. We shall discuss such results right after Corollaries 2.1, 2.2 and 2.3, respectively. Now we illustrate the statement of Theorem 2.3 in the three cases $p=1$, 4/3 and 2 that we have already singled out in the previous section. Namely, we have

$$\lim_{n\to\infty} n^{1/2}\,\frac{\|Q_n^U\|_1}{\|\beta_n^U\|_{3/2}^{3/2}}=\frac{2^{1/2}}{3^{1/2}\pi^{1/2}}=0.460658\ldots\quad a.s.,$$

$$\lim_{n\to\infty} n^{1/2}\,\frac{\|Q_n^U\|_{4/3}}{\|\beta_n^U\|_2^{3/2}}=\frac{2^{1/4}\{\Gamma(1/3)\}^{3/4}}{3^{5/4}\{\Gamma(2/3)\}^{3/4}}=0.502441\ldots\quad a.s.,$$

$$\lim_{n\to\infty} n^{1/2}\,\frac{\|Q_{n\,2}^U}{\|\beta_n^U\|_3^{3/2}}=\frac{1}{3^{1/2}}=0.577350\ldots\quad a.s.$$

We are now to derive from Theorem 2.3 the law of the iterated logarithm for $\|Q_n^U\|_p$. For this reason we recall the law of the iterated logarithm for $\|\beta_n^U\|_r$ of Csörgő and Shi (1998) that says that

$$\limsup_{n\to\infty}\frac{n^{1/2}}{(\log\log n)^{1/2}}\,\|\beta_n^U\|_r=\sqrt{2}\,c_3(r)\quad a.s. \tag{2.13}$$

for any $r\in[1,\infty)$, where $c_3(r)$ is defined in (1.6). In view of (2.13), Theorem 2.3 immediately implies the following corollary.

COROLLARY 2.1. *For any $p\in[1,\infty)$, we have*

$$\limsup_{n\to\infty}\frac{n^{5/4}}{(\log\log n)^{3/4}}\,\|Q_n^U\|_p\left[=\limsup_{n\to\infty}\frac{n^{1/4}}{(\log\log n)^{3/4}}\,\|nQ_n^U\|_p\right]$$
$$=\frac{2^{3/4}}{3^{1/2}}\,c_0(p)\,c_3^{3/2}(q)\quad a.s., \tag{2.14}$$

where $q:=3p/2$.

We call attention to the interesting fact that even the respective rates of convergence in (2.8) and (2.12) are the same, the corresponding rates of the laws of the iterated logarithm that we obtain from them (cf. (2.9) and (2.14)) are different. From the bracketed forms of the latter two statements we conclude that the almost sure rate of convergence in (2.14) is slower by $(\log\log n)^{1/2}$ than that in (2.9).

Again, the particular cases $p=1$, $4/3$ and 2 that we obtain from Corollary 2.1 may be of interest to a curious reader.

$$\limsup_{n\to\infty} \frac{n^{5/4}}{(\log\log n)^{3/4}} \|Q_n^U\|_1 = \frac{\{\Gamma(1/3)\}^{3/2}}{2^{1/4}3^{5/4}7^{1/4}\pi^{1/2}\{\Gamma(2/3)\}^3} = 0.130456\ldots \quad a.s.,$$

$$\limsup_{n\to\infty} \frac{n^{5/4}}{(\log\log n)^{3/4}} \|Q_n^U\|_{4/3} = \frac{2\{\Gamma(1/3)\}^{3/4}}{3^{5/4}\pi^{3/2}\{\Gamma(2/3)\}^{3/4}} = 0.151751\ldots \quad a.s.,$$

$$\limsup_{n\to\infty} \frac{n^{5/4}}{(\log\log n)^{3/4}} \|Q_n^U\|_2 = \frac{2^{1/4}3^{1/4}5^{1/4}\{\Gamma(2/3)\}^{3/2}}{\{\Gamma(1/3)\}^3} = 0.191812\ldots \quad a.s.$$

We are now to derive from Theorem 2.3 the "other" law of the iterated logarithm for $\|Q_n^U\|_p$. For this reason we recall the "other" law of the iterated logarithm for $\|\beta_n^U\|_r$ of Csörgő and Shi (1998) that states

$$\liminf_{n\to\infty} n^{1/2}(\log\log n)^{1/2} \|\beta_n^U\|_r = c_4(r) \quad a.s. \tag{2.15}$$

for any $r\in[1,\infty)$. Thus, in view of (2.15), Theorem 2.3 immediately implies the following corollary.

Corollary 2.2. *For any $p\in[1,\infty)$, we have*

$$\begin{aligned}\liminf_{n\to\infty} n^{5/4}(\log\log n)^{3/4} \|Q_n^U\|_p \, &\Big[= \liminf_{n\to\infty} n^{1/4}(\log\log n)^{3/4} \|nQ_n^U\|_p \Big] \\ &= \frac{1}{\sqrt{3}} c_0(p)\, c_4^{3/2}(q) \quad a.s.\end{aligned} \tag{2.16}$$

where $q := 3p/2$.

Comparing the two "other" laws of the iterated logarithm in (2.10) and (2.16) we note that, in the light of having already compared (2.8) and (2.14), this time around the almost sure rate of convergence is $(\log\log n)^{1/2}$ times faster in the case of $\|Q_n^U\|_p$ as in (2.16).

Since the only instance when we know the explicit value of the constant $c_4(q)$, $q := 3p/2$, in the range $p\geqslant 1$ is $c_4(2)$, we can therefore give the explicit value of the right-hand side of (2.16) only in the case $p=4/3$. This is one of the reasons of singling out the case $p=4/3$ in our previous considerations. Hence, we have

$$\liminf_{n\to\infty} n^{5/4}(\log\log n)^{3/4}\ \|Q_n^U\|_{4/3}=\frac{\{\Gamma(1/3)\}^{3/4}}{2^2 3^{5/4}\{\Gamma(2/3)\}^{3/4}}=0.105625... \quad a.s.$$

We are now to consider convergence in distribution of $\|Q_n^U\|_p$. A fundamental result of Komlós, Major and Tusnády [KMT] (1975) states that on a possibly enlarged probability space there normalized sequence $\{\sqrt{n}\,\beta_n^U\}$ of empirical process can be approximated by a sequence $\{\mathscr{B}_n\}$ of Brownian bridges in such a way that

$$\|\sqrt{n}\,\beta_n^U-\mathscr{B}_n\|_\infty=\mathcal{O}\left(\frac{\log n}{\sqrt{n}}\right) \quad a.s. \tag{2.17}$$

In view of (2.17), Theorem 2.3 obviously implies the following corollary.

COROLLARY 2.3. *For any $p\in[1,\infty)$, we have*

$$\begin{aligned}\lim_{n\to\infty}\mathbf{P}\{n^{5/4}\ \|Q_n^U\|_p\leqslant x\}&\left[=\lim_{n\to\infty}\mathbf{P}\{n^{1/4}\ \|nQ_n^U\|_p\leqslant x\}\right]\\&=\mathbf{P}\left\{\frac{1}{\sqrt{3}}\,c_0(p)\ \|\mathscr{B}\|_q^{3/2}\leqslant x\right\}, \qquad x>0,\end{aligned} \tag{2.18}$$

where $q:=3p/2$.

In particular, when $p=4/3$, Corollary 2.3 and representation (1.13) for $\mathbf{P}\{\|\mathscr{B}\|_2\leqslant x\}$ immediately imply

$$\begin{aligned}&\lim_{n\to\infty}\mathbf{P}\{n^{5/4}\ \|Q_n^U\|_{4/3}\leqslant x\}\\&\quad=1+\frac{2}{\pi}\sum_{k=1}^{\infty}(-1)^k\int_{(2k-1)\pi}^{2k\pi}\frac{1}{\sqrt{-t\sin t}}\\&\qquad\times\exp\left\{-\frac{3^{5/3}\Gamma(2/3)}{2^{4/3}\Gamma(1/3)}\,t^2x^{4/3}\right\}dt, \qquad x>0.\end{aligned} \tag{2.19}$$

We note in passing that, analogously to the case of (2.16), again $p=4/3$ is the only one among all the cases $p\geqslant 1$ that has an explicit expression, this time around for the limiting distribution of $n^{5/4}\ \|Q_n^U\|_p$. Naturally, one may also wish to have an analog of statement (2.19) in the Hilbertian $p=2$ case. However, this requires an explicit formula for the distribution of $\|\mathscr{B}\|_3$ which, to the best of our knowledge, has not yet been obtained.

We conclude this section with two remarks.

Remark 2.1. It is very likely that the analog of statement (2.12) in the sup-norm is of the following form

$$\lim_{n\to\infty} \mathfrak{a}_n \frac{\|Q_n^U\|_\infty}{\|\beta_n^U\|_\infty^{3/2}} = \mathfrak{c} \quad a.s., \tag{2.20}$$

where $\mathfrak{a}_n$ are normalizing constants that tend to infinity slower than $n^{1/2}$ (cf. (2.12)), and $\mathfrak{c} \in (0, \infty)$. Hence, the only guess that we can make right now about $\mathfrak{a}_n$ is of the form

$$\mathfrak{a}_n := \mathfrak{b}_n n^{1/2},$$

where $n \mapsto \mathfrak{b}_n$ is a slowly varying function converging to 0 when $n \to \infty$. Moreover, by an almost identical reasoning to that given in Subsection 3.1 of Section 3 below, one can show that statement (2.20) is equivalent (on a probably enlarged probability space) to the following one

$$\lim_{n\to\infty} \frac{\mathfrak{b}_n n^{1/4}}{\|\mathscr{B}_n\|_\infty^{3/2}} \left\| \mathscr{B}_n(\cdot) \int_0^1 \left\{ \mathscr{B}_n\left(\cdot - \frac{s}{\sqrt{n}} \mathscr{B}_n(\cdot)\right) - \mathscr{B}_n(\cdot) \right\} ds \right\|_\infty = \tilde{c} \quad a.s. \tag{2.21}$$

with the Brownian bridge $\mathscr{B}_n$ of (3.32). In any case, the normalizing constant $\mathfrak{a}_n$ of (2.20) must be different from that $n^{1/2}$ of (2.12). Hence, we make the following conjecture.

CONJECTURE 2.1. *The statement*

$$a_n Q_n^U \to_d Y, \qquad n \to \infty,$$

cannot hold true in the space $D[0, 1]$ for any sequence $\{a_n\}$ of positive real numbers and for any non-degenerate random element Y of the space $D[0, 1]$.

In turn, if Conjecture 2.1 were to be true, then we would have yet another example, i.e., Q_n^U, when a process itself would not converge in distribution to a nondegenerate process but norms of it would (cf. Theorem 1.1).

Remark 2.2. In the literature we also find the general Vervaat process V_n which is a generalization of the above considered [(0, 1)-uniform] Vervaat process V_n^U. It is obvious that the results of this paper can be generalized in such a way that they would cover the general Vervaat process V_n as well. However, it seems that a solution of this problem under reasonably optimal assumptions may constitute a rather challenging mathematical task which is definitely not within the scope of the present paper. We also note that the [general] Vervaat process V_n first appeared and was put to good use in Csörgő and Zitikis (1996). We refer to Zitikis (1998) for the only survey paper on this subject so far, as well as to Horváth (1983) for a similar ideas that are related to the Vervaat process V_n. This work was overlooked by, and thus missing from, Zitikis (1998). For related though rather different limit theorems for the general Vervaat process V_n, we refer to Csörgő and Zitikis (1999).

3. PROOF OF THEOREM 2.3

We now briefly overview the five following subsections in order to get the main idea of the proof of Theorem 2.3.

In Subsection 3.1, using the strong approximation methodology, we simplify and ultimately convert statement (2.12) of Theorem 2.3 into a statement concerning a Kiefer process. In Subsection 3.2 we replace the Kiefer process by a sequence of Brownian bridges. In order to gain some additional and useful independence structure between some of the random processes involved, in Subsection 3.3 we replace the Brownian bridges of Subsection 3.2 by another sequence of Brownian bridges. Subsection 3.4 is somewhat technical. It employs, in a most crucial way, conditional probabilities together with the independence structure obtained in Subsection 3.3 and in this way concludes the proof of Theorem 2.3 under the assumption that a certain bound, i.e., (3.71), holds true. The proof of the just mentioned bound, i.e., (3.71), is given in Subsection 3.5, where we replace the Brownian bridges of Subsection 3.3 by Wiener processes. This we do for the sake of using convenient properties of the Wiener processes that play a crucial role in our proof.

Remark 3.1. When dealing with, for example, the Brownian bridges $\mapsto \mathscr{B}_n(t)$ (that are defined on the interval $[0, 1]$), in the course of proof we may occasionally go outside the interval $[0, 1]$ and in this way end up with (undefined) quantities like $\mathscr{B}_n(z)$ for some z to the left of 0 or to the right of 1. This situation can easily be fixed by bringing in independent Brownian bridges defined on, say, $[-1, 0]$ and $[1, 2]$. However, these technical adjustments do not influence any of our results, and we have therefore removed further related discussions from the proofs making them less tedious. The same remark applies to other processes as well.

3.1. *From the Vervaat Process to a Kiefer Process*

The fact that $V_n^U(t)$ is the volume of area B (cf. Fig. 1.1) immediately implies the following representation

$$V_n^U(t) = \int_{E_n(t)}^{t} \{E_n^{-1}(s) - t\}\, ds \tag{3.1}$$

for $V_n^U(t)$. We now interrupt the proof of Theorem 2.3 for a couple of remarks concerning (3.1).

Remark 3.2. Vervaat (1972a,b) finds it more convenient to work with the following representation

$$V_n^U(t) = \int_{E_n^{-1}(t)}^{t} \{E_n(s) - t\}\, ds \tag{3.2}$$

instead of (3.1). In the context of the present paper, however, we have found representation (3.1) to be more convenient to work with. We conclude the remark by noting that representation (3.2), just like (3.1), follows immediately from the representation of $V_n^U(t)$ as the volume of area B (cf. Fig. 1.1). The only difference between the derivations of (3.1) and (3.2) is that in each case we integrate with respect to the Lebesgue measure acting on different coordinate axes.

Remark 3.3. For those who are not entirely satisfied with the "geometrical" proof of (3.1) via Fig. 1.1, we note that representation (3.1) can easily be proved by subdividing the interval $(0, 1]$ into the $n+1$ subintervals $[0, U_{1:n})$ $[U_{1:n}, U_{2:n})$, ..., $(U_{n:n}, 1)$ and then checking the equality (3.1) on each subinterval separately.

We now resume the proof of Theorem 2.3. Starting with representation (3.1), we obtain the following three elementary equalities

$$\begin{aligned} V_n^U(t) &= \int_{E_n(t)}^{t} \{E_n^{-1}(s) - s\} + \{s - E_n(t)\} + \{E_n(t) - t\}\, ds \\ &= \int_{E_n(t)}^{t} \{\gamma_n^U(s) + \beta_n^U(t)\}\, ds + \int_{E_n(t)}^{t} \{s - E_n(t)\}\, ds \\ &= \int_{E_n(t)}^{t} \{\gamma_n^U(s) + \beta_n^U(t)\}\, ds + \tfrac{1}{2}\{\beta_n^U(t)\}^2. \end{aligned} \tag{3.3}$$

The last equality of (3.3) shows that

$$Q_n^U(t) = \int_{t+\beta_n^U(t)}^{t} \{\gamma_n^U(s) + \beta_n^U(t)\}\, ds. \tag{3.4}$$

We are now to reduce the right-hand side of (3.4) to an easier object that involves only the empirical process β_n^U. In other words, we are to replace the quantile process γ_n^U on the right-hand side of (3.4) by β_n^U in such a way that the replacement will not affect the result (2.12) that we want to prove for $Q_n^U(t)$. We start off with the equality

$$\gamma_n^U(j/n) = -\beta_n^U(U_{j:n}) \tag{3.5}$$

that holds true for all $j = 0, 1, ..., n$, provided that we agree on the notation $E_n^{-1}(0) := E_n^{-1}(0+)$, which we do. From (3.5) we immediately obtain that

$$\begin{aligned} \gamma_n^U(s) &= -\beta_n^U(E_n^{-1}(s)) + \varepsilon_n(s) \\ &= -\beta_n^U(s + \gamma_n^U(s)) + \varepsilon_n(s), \end{aligned} \tag{3.6}$$

where the remainder term ε_n is obviously such that $\|\varepsilon_n\|_\infty \leqslant n^{-1}$. Using now representation (3.6) on the right-hand side of equality (3.4) several times, we obtain the following two equalities

$$\begin{aligned} Q_n^U(t) &= \int_{t+\beta_n^U(t)}^{t} \{-\beta_n^U(s + \gamma_n^U(s)) + \beta_n^U(t)\}\, ds + \int_{t+\beta_n^U(t)}^{t} \varepsilon_n(s)\, ds \\ &= \int_{t+\beta_n^U(t)}^{t} \{-\beta_n^U(s - \beta_n^U(s)) + \beta_n^U(t)\}\, ds \\ &\quad + \int_{t+\beta_n^U(t)}^{t} \{-\beta_n^U(s - \beta_n^U(s) + [\beta_n^U(s) - \beta_n^U(s + \gamma_n^U(s))] + \varepsilon_n(s)) \\ &\quad + \beta_n^U(s - \beta_n^U(s))\}\, ds + \int_{t+\beta_n^U(t)}^{t} \varepsilon_n(s)\, ds \\ &=: \Lambda_n^*(t) + \varepsilon_n^*(t) + \varepsilon_n^{**}(t). \end{aligned} \tag{3.7}$$

We are now to bound $\|\varepsilon_n^{**}\|_\infty$. The bound $\|\varepsilon_n\|_\infty \leqslant n^{-1}$ and the law of the iterated logarithm

$$\limsup_{n\to\infty} \sqrt{\frac{n}{\log\log n}}\, \|\beta_n^U\|_\infty = \frac{1}{\sqrt{2}} \quad a.s. \tag{3.8}$$

(due to Smirnov, 1944) immediately imply the bound

$$\|\varepsilon_n^{**}\|_\infty \leqslant \frac{1}{n^{3/2}}\{n^\epsilon + o_{a.s.}(1)\} \tag{3.9}$$

for any fixed $\epsilon > 0$. In turn, bound (3.9) and the "other" law of the iterated logarithm

$$\liminf_{n\to\infty} \sqrt{n \log\log n}\, \|\beta_n^U\|_q = c_4(q) \quad a.s. \tag{3.10}$$

(due to Csörgő and Shi, 1998, cf. Donsker and Varadhan, 1977) imply

$$\limsup_{n\to\infty} \sqrt{n}\, \frac{\|\varepsilon_n^{**}\|_p}{\|\beta_n^U\|_q^{3/2}} = 0 \quad a.s. \tag{3.11}$$

Thus, the process ε_n^{**} is too asymptotically small to influence statement (2.12).

The proof that the process ε_n^* is also too asymptotically small to influence statement (2.12) is a little bit more involved. To start off we recall the definition

$$\omega_n(h) := \sup_{\substack{t,\, s \in (0,\,1) \\ |t-s| \leqslant h}} |\beta_n^U(t) - \beta_n^U(s)| \tag{3.12}$$

of the oscillation modulus of the empirical process β_n^U, which immediately gives us the first of the following two bounds

$$\begin{aligned} \|\varepsilon_n^*\|_\infty &\leqslant \|\beta_n^U\|_\infty\, \omega_n(\omega_n(\|\gamma_n^U\|_\infty) + \|\varepsilon_n\|_\infty) \\ &\leqslant \|\beta_n^U\|_\infty\, \omega_n(\omega_n(\|\beta_n^U\|_\infty) + n^{-1}), \end{aligned} \tag{3.13}$$

whereas the second bound of (3.13) follows from the equality $\|\gamma_n^U\|_\infty = \|\beta_n^U\|_\infty$ and bound $\|\varepsilon_n\|_\infty \leqslant n^{-1}$. To estimate the right-hand side of (3.13), it is the easiest way for us to recall the well-known result of W. Stute saying that

$$\lim_{n\to\infty} \sqrt{\frac{n}{a_n \log(1/a_n)}}\, \omega_n(a_n) = \sqrt{2} \quad a.s. \tag{3.14}$$

for any non-increasing sequence $\{a_n\}$ of positive numbers such that the sequence $\{na_n\}$ is non-decreasing, the sequence $\{\log(1/a_n)/(\log\log n)\}$

converges to ∞, and the sequence $\{\log(1/a_n)/(na_n)\}$ converges to 0. When applied on the right-hand side of (3.13), statement (3.14) implies the bound

$$\|\varepsilon_n^*\|_\infty \leqslant \frac{1}{n^{11/8}}\{n^\epsilon + o_{a.s.}(1)\} \tag{3.15}$$

that holds true for any fixed $\epsilon > 0$. Taking a sufficiently small $\epsilon > 0$ in (3.15), we obtain from bound (3.15) and the "other" law of the iterated logarithm (3.10) for $\|\beta_n^U\|_q$ that the statement

$$\limsup_{n\to\infty} \sqrt{n}\,\frac{\|\varepsilon_n^*\|_p}{\|\beta_n^U\|_q^{3/2}} = 0 \quad a.s. \tag{3.16}$$

holds true.

In view of statements (3.11) and (3.16) for ε_n^{**} and ε_n^*, respectively, and representation (3.7), we obtain the statement of Theorem 2.3 by demonstrating the validity of the following one

$$\lim_{n\to\infty} \sqrt{n}\,\frac{\|\Lambda_n^*\|_p}{\|\beta_n^U\|_q^{3/2}} = \frac{1}{\sqrt{3}}\,c_0(p) \quad a.s. \tag{3.17}$$

The proof of (3.17) starts with the following elementary equality

$$\Lambda_n^*(t) = \beta_n^U(t)\int_0^1 \{\beta_n^U(t + s\beta_n^U(t) - \beta_n^U(t + s\beta_n^U(t))) - \beta_n^U(t)\}\,ds \tag{3.18}$$

that follows immediately from the definition of Λ_n^* by changing the variable of integration s into $t + s\beta_n^U(t)$. The process Λ_n^* can further be simplified by deleting one β_n^U from the right-hand side of (3.18) in the following way

$$\begin{aligned}
\Lambda_n^*(t) &= \beta_n^U(t)\int_0^1 \{\beta_n^U(t + (s-1)\,\beta_n^U(t)) - \beta_n^U(t)\}\,ds \\
&\quad + \beta_n^U(t)\int_0^1 \Big\{\beta_n^U(t + (s-1)\,\beta_n^U(t) + [\beta_n^U(t) - \beta_n^U(t + s\beta_n^U(t))]) \\
&\quad - \beta_n^U(t + (s-1)\,\beta_n^U(t))\Big\}\,ds \\
&=: \Lambda_n^{**}(t) + \varepsilon_n^\diamond(t),
\end{aligned} \tag{3.19}$$

where the remainder process $\varepsilon_n^\diamond$ appears to be asymptotically smaller than the process Λ_n^{**}. Indeed, from the asymptotic point of view the process $\varepsilon_n^\diamond$

is similar to ε_n^* of (3.7). Consequently, in an almost identical way statement (3.16) was proved for ε_n^*, one now easily checks the validity of the statement

$$\limsup_{n\to\infty} \sqrt{n}\,\frac{\|\varepsilon_n^{\diamond}\|_p}{\|\beta_n^U\|_q^{3/2}}=0 \quad a.s. \tag{3.20}$$

Thus, by statement (3.20) and representation (3.19) we have reduced the proof of Theorem 2.3 to showing that

$$\lim_{n\to\infty} \sqrt{n}\,\frac{\|\Lambda_n^{**}\|_p}{\|\beta_n^U\|_q^{3/2}}=\frac{1}{\sqrt{3}}\,c_0(p) \quad a.s. \tag{3.21}$$

holds true.

We start the proof of (3.21) with the elementary equality

$$\Lambda_n^{**}(t)=\beta_n^U(t)\int_0^1 \{\beta_n^U(t-s\beta_n^U(t))-\beta_n^U(t)\}\,ds \tag{3.22}$$

The right-hand side of (3.22) involves only the empirical process β_n^U. Therefore, on the right-hand side of (3.22), we can now directly apply the fundamental result of KMT (1975) saying that, on a possibly larger probability space, the representation

$$\beta_n^U(t)=\frac{1}{n}\,\mathscr{K}_n(t)+\varepsilon_n^{\diamond\diamond}(t), \tag{3.23}$$

holds true with a certain Kiefer process

$$\mathscr{K}:(t,n)\mapsto \mathscr{K}_n(t),$$

where the remainder process $\varepsilon_n^{\diamond\diamond}$ is such that

$$\|\varepsilon_n^{\diamond\diamond}\| \leqslant \frac{(\log n)^2}{n}\,\{c+o_{a.s.}(1)\} \tag{3.24}$$

for a universal constant $c>0$ (cf. Csörgő and Szyszkowicz, 1998, for further mathematical and historical details on the subject). In view of bound (3.24) one immediately notices via representations (3.22) and (3.23) that the process Λ_n^{**} is asymptotically equivalent to the process $\Lambda_n^{\diamond\diamond}$ defined as follows

$$\Lambda_n^{\diamond\diamond}(t)=\frac{1}{n^2}\,\mathscr{K}_n(t)\int_0^1 \left\{\mathscr{K}_n\left(t-\frac{s}{n}\,\mathscr{K}_n(t)\right)-\mathscr{K}_n(t)\right\} ds. \tag{3.25}$$

Indeed, we are now to demonstrate that

$$\limsup_{n\to\infty} \sqrt{n}\, \frac{\|\Lambda_n^{**} - \Lambda_n^{\diamond\diamond}\|_p}{\|\beta_n^U\|_q^{3/2}} = 0 \quad a.s. \tag{3.26}$$

We start the proof of (3.26) by noting that the "other" law of the iterated logarithm (3.10) for $\|\beta_n^U\|_q$ implies statement (3.26), provided that

$$\limsup_{n\to\infty} n^{5/4}(\log\log n)^{3/4} \, \|\Lambda_n^{**} - \Lambda_n^{\diamond\diamond}\|_p = 0 \quad a.s. \tag{3.27}$$

In order to prove (3.27), we use representation (3.23) and rewrite the difference $\Lambda_n^{**} - \Lambda_n^{\diamond\diamond}$ in the following way

$$\begin{aligned}
\Lambda_n^{**}(t) - \Lambda_n^{\diamond\diamond}(t) = {} & \varepsilon_n^{\diamond\diamond}(t) \int_0^1 \{\beta_n^U(t - s\beta_n^U(t)) - \beta_n^U(t)\}\, ds \\
& + \frac{1}{n}\, \mathcal{K}_n(t) \int_0^1 \left\{ \beta_n^U\left(t - \frac{s}{n}\, \mathcal{K}_n(t) - s\varepsilon_n^{\diamond\diamond}(t)\right)\right. \\
& \left. - \beta_n^U\left(t - \frac{s}{n}\, \mathcal{K}_n(t)\right)\right\} ds \\
& + \frac{1}{n}\, \mathcal{K}_n(t) \int_0^1 \varepsilon_n^{\diamond\diamond}\left(t - \frac{s}{n}\, \mathcal{K}_n(t)\right) ds \\
& + \frac{1}{n}\, \mathcal{K}_n(t)\, \varepsilon_n^{\diamond\diamond}(t).
\end{aligned} \tag{3.28}$$

Furthermore, using the following four tools

1. bound (3.24) for $\|\varepsilon_n^{\diamond\diamond}\|_\infty$,
2. Smirnov's law of the iterated logarithm (3.8) for $\|\beta_n^U\|_\infty$,
3. the following easy consequence

$$\limsup_{n\to\infty} \frac{\|\mathcal{K}_n\|_\infty}{\sqrt{n\log\log n}} = \frac{1}{\sqrt{2}} \quad a.s. \tag{3.29}$$

of (3.8) and (3.23)–(3.24),

4. the aforementioned W. Stute's result (3.14),

we immediately obtain that the sup-norm $\|\cdot\|_\infty$ of all the four summands on the right-hand side of (3.28) converge to 0 a.s. with the rate at least as fast as $n^{-6/4+\epsilon}$ for any fixed $\epsilon > 0$. This obviously implies that statement (3.27) holds true. Consequently, Theorem 2.3 follows provided that the statement

$$\lim_{n\to\infty} \sqrt{n}\, \frac{\|\Lambda_n^{\diamond\diamond}\|_p}{\|\beta_n^U\|_q^{3/2}} = \frac{1}{\sqrt{3}}\, c_0(p) \quad a.s. \tag{3.30}$$

holds true. Furthermore, using representation (3.23) and bound (3.24), we easily replace $\|\beta_n^U\|_q$ on the right-hand side of (3.30) by $n^{-1}\,\|\mathscr{K}\|_q$ and in this way reduce the statement of (3.30) to the following one

$$\lim_{n\to\infty} \frac{1}{\|\mathscr{K}_n\|_q^q} \int_0^1 \left| \mathscr{K}_n(t) \int_0^1 \left\{ \mathscr{K}_n\left(t - \frac{s}{n}\, \mathscr{K}_n(t)\right) - \mathscr{K}_n(t) \right\} ds \right|^p dt$$
$$= \frac{1}{3^{p/2}}\, c_0^p(p) \quad a.s. \tag{3.31}$$

As we obviously see, statement (3.31) is completely based on the Kiefer processes $\mathscr{K}$. On this note we complete the current subsection, starting the proof of (3.31) in next Subsection 3.2.

3.2. *From the Kiefer Process $\mathscr{K}$ to a Brownian Bridge*

We ought to start this subsection with the following precautionary remark.

Remark 3.4. Statement (3.31) can easily be reformulate into another, equivalent statement involving only a Brownian bridge instead of $\mathscr{K}_n$. To see this, we set

$$\mathscr{B}_n := \frac{1}{\sqrt{n}}\, \mathscr{K}_n.$$

Obviously, $\mathscr{B}_n$ is a Brownian bridge for any fixed $n \in \mathbf{N}$. Statement (3.31) can now be rewritten in the following, equivalent way

$$\lim_{n\to\infty} n^{p/4} \frac{1}{\|\mathscr{B}_n\|_q^q} \int_0^1 \left| \mathscr{B}_n(t) \int_0^1 \left\{ \mathscr{B}_n\left(t - \frac{s}{\sqrt{n}}\, \mathscr{B}_n(t)\right) - \mathscr{B}_n(t) \right\} ds \right|^p dt$$
$$= \frac{1}{3^{p/2}}\, c_0^p(p) \quad a.s. \tag{3.32}$$

However, our goal in this subsection is somewhat more subtle than just statement (3.32), even though the title of this subsection may suggest that (3.32) is what we want to have.

We continue the proof of statement (3.31) with the following representation

$$
\begin{aligned}
&\int_0^1 \left| \mathcal{K}_n(t) \int_0^1 \left\{ \mathcal{K}_n\left(t-\frac{s}{n}\,\mathcal{K}_n(t)\right)-\mathcal{K}_n(t)\right\} ds \right|^p dt \\
&\quad =\sum_{i=0}^{N-1} \int_{i/N}^{(i+1)/N} |\mathcal{K}_n(t)|^p \left| \int_0^1 \left\{ \mathcal{K}_n\left(t-\frac{s}{n}\,\mathcal{K}_n(t)\right)-\mathcal{K}_n(t)\right\} ds \right|^p dt \\
&\quad =\sum_{i=0}^{N-1} \int_{i/N}^{(i+1)\,N} \left\{ |\mathcal{K}_n(t)|^p - \left|\mathcal{K}_n\left(\frac{i}{N}\right)\right|^p \right\} \left| \int_0^1 \left\{ \mathcal{K}_n\left(t-\frac{s}{n}\,\mathcal{K}_n(t)\right) \right.\right. \\
&\qquad \left.\left. -\mathcal{K}_n(t)\right\} ds \right|^p dt \\
&\qquad +\sum_{i=0}^{N-1} \int_{i/N}^{(i+1)/N} \left|\mathcal{K}_n\left(\frac{i}{N}\right)\right|^p \left\{ \left| \int_0^1 \left\{ \mathcal{K}_n\left(t-\frac{s}{N}\,\mathcal{K}_n(t)\right)-\mathcal{K}_n(t)\right\} ds \right|^p \right. \\
&\qquad \left. -\left| \int_0^1 \left\{ \mathcal{K}_n\left(t-\frac{s}{n}\,\mathcal{K}_n\left(\frac{i}{N}\right)\right)-\mathcal{K}_n(t)\right\} ds \right|^p \right\} dt \\
&\qquad +\sum_{i=0}^{N-1} \int_{i/N}^{(i+1)/N} \left|\mathcal{K}_n\left(\frac{i}{N}\right)\right|^p \left| \int_0^1 \left\{ \mathcal{K}_n\left(t-\frac{s}{N}\,\mathcal{K}_n\left(\frac{i}{N}\right)\right)-\mathcal{K}_n(t)\right\} ds \right|^p dt \\
&\quad =: \delta^*_{n,N}+\delta^{**}_{n,N}+\Psi_{n,N}, \qquad (3.33)
\end{aligned}
$$

where the parameter $N \in \mathbf{N}$ can be an arbitrary number. We choose, however, N as follows

$$N := \text{ the integer part of } (n^{\epsilon_0}), \tag{3.34}$$

where $\epsilon_0 > 0$ is a small constant to be specified below.

We are now to prove the statement

$$\limsup_{n\to\infty} \frac{|\delta^*_{n,N}|}{\|\mathcal{K}_n\|_q^q} = 0 \quad a.s. \tag{3.35}$$

that will indicate that $\delta^*_{n,N}$ is too asymptotically small to influence statement (3.31). Denoting the oscillation modulus of $\mathcal{K}_n$ by

$$\Omega_n(h) := \sup_{\substack{t,\, s \in (0,1) \\ |t-s| \leqslant h}} |\mathcal{K}_n(t)-\mathcal{K}_n(s)|$$

we obtain the following bound

$$\left| \int_0^1 \left\{ \mathcal{K}_n\left(t-\frac{s}{n}\,\mathcal{K}_n(t)\right)-\mathcal{K}_n(t)\right\} ds \right|^p \leqslant \Omega_n^p\left(\frac{1}{n}\,\|\mathcal{K}_n\|_\infty\right) \tag{3.36}$$

A well-known result of W. Stute says that

$$\lim_{n\to\infty} \frac{\Omega_n(a_n)}{\sqrt{na_n \log(1/a_n)}} = \sqrt{2} \quad a.s. \tag{3.37}$$

for any non-increasing sequence $\{a_n\}$ of positive numbers such that the sequence $\{na_n\}$ is non-decreasing and the sequence $\{\log(1/a_n)/(\log\log n)\}$ converges to ∞. Statement (3.37) and the law of the iterated logarithm (3.29) for $\|\mathscr{K}_n\|_\infty$, imply via (3.36) that

$$\left|\int_0^1 \left\{\mathscr{K}_n\left(t - \frac{s}{n}\mathscr{K}_n(t)\right) - \mathscr{K}_n(t)\right\} ds\right|^p \leqslant n^{p/4}\{n^\epsilon + o_{a.s.}(1)\} \tag{3.38}$$

for any fixed $\epsilon > 0$. To proceed with the estimation of $|\delta^*_{n,N}|$, we now estimate the distance between $|\mathscr{K}_n(t)|^p$ and $|\mathscr{K}_n(i/N)|^p$. For this reason we use the inequality

$$|x^r - y^r| \leqslant c\,|x-y|^r + cx^{r-1}\,|x-y|, \qquad x \geqslant 0, \quad y \geqslant 0, \tag{3.39}$$

that holds true for any $r \geqslant 1$ and some constant c depending only on r. We obtain that, for any $t \in [i/N, (i+1)/N]$,

$$\begin{aligned}
&|\,|\mathscr{K}_n(t)|^p - |\mathscr{K}_n(i/N)|^p\,| \\
&\quad \leqslant c\,|\mathscr{K}_n(t) - \mathscr{K}_n(i/N)|^p + c\,|\mathscr{K}_n(t)|^{p-1}\,|\mathscr{K}_n(t) - \mathscr{K}_n(i/N)| \\
&\quad \leqslant c\,\Omega_n^p(1/N) + c\,|\mathscr{K}_n(t)|^{p-1}\,\Omega_n(1/N) \\
&\quad \leqslant \left\{\frac{n}{N}\right\}^{p/2}\{n^\epsilon + o_{a.s.}(1)\} + n^{(p-1)/2}\left\{\frac{n}{N}\right\}^{1/2}\{n^\epsilon + o_{a.s.}(1)\} \\
&\quad \leqslant \frac{n^{p/2}}{N^{1/2}}\{n^\epsilon N^\epsilon + o_{a.s.}(1)\}
\end{aligned} \tag{3.40}$$

for any fixed $\epsilon > 0$. Taking bounds (3.38) and (3.40) together, we get the following one

$$|\delta^*_{n,N}| \leqslant \frac{n^{3p/4}}{N^{1/2}}\{n^\epsilon N^\epsilon + o_{a.s.}(1)\} \tag{3.41}$$

for any fixed $\epsilon > 0$. On the other hand, by the "other" law of the iterated logarithm

$$\liminf_{n\to\infty} \sqrt{\frac{\log\log n}{n}}\,\|\mathscr{K}_n\|_q = c_4(q) \quad a.s. \tag{3.42}$$

that follows from (3.10) and (3.23)–(3.24), we obtain the bound

$$\|\mathcal{K}_n\|_q^q \geqslant n^{3p/4}\{n^{-\epsilon} + o_{a.s.}(1)\} \tag{3.43}$$

for any fixed $\epsilon > 0$. Bounds (3.41) and (3.43) taken together imply that

$$\limsup_{n\to\infty} \frac{|\delta^*_{n,N}|}{\|\mathcal{K}_n\|_q^q} \leqslant \frac{1}{N^{1/2}} \{n^\epsilon N^\epsilon + o_{a.s.}(1)\} \tag{3.44}$$

for any fixed $\epsilon > 0$. By taking $\epsilon > 0$ sufficiently small, we immediately derive from (3.44) that statement (3.35) holds true.

We are now to prove the statement

$$\limsup_{n\to\infty} \frac{|\delta^{**}_{n,N}|}{\|\mathcal{K}_n\|_q^q} = 0 \quad a.s. \tag{3.45}$$

Due to the law of the iterated logarithm (3.29) for $\mathcal{K}_n$, we have the second inequality of the following two ones

$$|\mathcal{K}_n(i/N)|^p \leqslant \|\mathcal{K}_n\|_\infty^p \leqslant n^{p/2}\{n^\epsilon + o_{a.s.}(1)\}, \tag{3.46}$$

where $\epsilon > 0$ is any fixed number, whereas the first inequality is trivial. Next, due to bound (3.39), we have that, for any $t \in [i/N, (i+1)/N]$,

$$\begin{aligned}
&\Bigg| \left| \int_0^1 \left\{ \mathcal{K}_n\left(t - \frac{s}{n}\mathcal{K}_n(t)\right) - \mathcal{K}_n(t) \right\} ds \right|^p \\
&\qquad - \left| \int_0^1 \left\{ \mathcal{K}_n\left(t - \frac{s}{n}\mathcal{K}_n\left(\frac{i}{N}\right)\right) - \mathcal{K}_n(t) \right\} ds \right|^p \Bigg| \\
&\quad \leqslant c \left| \int_0^1 \left\{ \mathcal{K}_n\left(t - \frac{s}{n}\mathcal{K}_n(t)\right) - \mathcal{K}_n\left(t - \frac{s}{n}\mathcal{K}_n\left(\frac{i}{N}\right)\right) \right\} ds \right|^p \\
&\qquad + c \left| \int_0^1 \left\{ \mathcal{K}_n\left(t - \frac{s}{n}\mathcal{K}_n(t)\right) - \mathcal{K}_n(t) \right\} ds \right|^{p-1} \\
&\qquad \times \left| \int_0^1 \left\{ \mathcal{K}_n\left(t - \frac{s}{n}\mathcal{K}_n(t)\right) - \mathcal{K}_n\left(t - \frac{s}{n}\mathcal{K}_n\left(\frac{i}{N}\right)\right) \right\} ds \right| \\
&\quad \leqslant c\Omega_n^p\left(\frac{1}{n}\Omega_n\left(\frac{1}{N}\right)\right) + c\Omega_n^{p-1}\left(\frac{1}{n}\|\mathcal{K}_n\|_\infty\right)\Omega_n\left(\frac{1}{n}\Omega_n\left(\frac{1}{N}\right)\right).
\end{aligned} \tag{3.47}$$

An application of the law of the iterated logarithm (3.29) for $\|\mathscr{K}_n\|_\infty$ and W. Stute's result (3.37) imply that the right-hand side of (3.47) does not exceed

$$\frac{n^{p/4}}{N^{1/4}}\{n^\epsilon N^\epsilon + o_{a.s.}(1)\}$$

for any fixed $\epsilon > 0$. This fact, together with previously proved bounds (3.46) and (3.43), implies that

$$\limsup_{n\to\infty} \frac{|\delta^*_{n,N}|}{\|\mathscr{K}_n\|_q^q} \leqslant \limsup_{n\to\infty} \frac{1}{N^{1/4}}\{n^\epsilon N^\epsilon + o_{a.s.}(1)\} \quad a.s. \tag{3.48}$$

By choosing a sufficiently small $\epsilon > 0$, we make the right-hand side of (3.48) equal to 0 a.s. This completes the proof of statement (3.45).

Using now both statements (3.35) and (3.45) in representation (3.33), we immediately conclude that statement (3.31) holds true provided that

$$\lim_{n\to\infty} \frac{\Psi_{n,N}}{\|\mathscr{K}_n\|_q^q} = \frac{1}{3^{p/2}}\, c_0^p(p) \quad a.s. \tag{3.49}$$

We now rewrite the denominator $\|\mathscr{K}_n\|_q^q$ on the left-hand side of (3.49) in the following way

$$\begin{aligned}\|\mathscr{K}_n\|_q^q &= \frac{1}{N}\sum_{i=0}^{N-1}\left|\mathscr{K}_n\left(\frac{i}{N}\right)\right|^q + \sum_{i=0}^{N-1}\int_{i/N}^{(i+1)/N}\left\{|\mathscr{K}_n(t)|^q - \left|\mathscr{K}_n\left(\frac{i}{N}\right)\right|^q\right\}dt \\ &=: \frac{1}{N}\sum_{i=0}^{N-1}\left|\mathscr{K}_n\left(\frac{i}{N}\right)\right|^q + \delta_n^{\circ\circ}.\end{aligned} \tag{3.50}$$

In view of bound (3.40) with q instead of p, the remainder term $\delta_n^{\circ\circ}$ is obviously such that

$$|\delta_n^{\circ\circ}| \leqslant \frac{n^{3p/4}}{N^{1/2}}\{n^\epsilon N^\epsilon + o_{a.s.}(1)\}. \tag{3.51}$$

In view of bounds (3.51) and (3.43), and by choosing $\epsilon > 0$ in (3.51) sufficiently small, we immediately conclude that statement (3.49) is a consequence of the following one

$$\lim_{n\to\infty} \frac{N\Psi_{n,N}}{\sum_{i=0}^{N-1}|\mathscr{K}_n(i/N)|^q} = \frac{1}{3^{p/2}}\, c_0^p(p) \quad a.s. \tag{3.52}$$

Using the above introduced notation $\mathscr{B}_n := n^{-1/2}\mathscr{K}_n$, we rewrite $\Psi_{n,N}$ as follows

$$\begin{aligned}
&\Psi_{n,N}\\
&= n^p \sum_{i=0}^{N-1} \left|\mathscr{B}_n\left(\frac{i}{N}\right)\right|^p \int_{i/N}^{(i+1)/N} \left|\int_0^1 \left\{\mathscr{B}_n\left(t - \frac{s}{\sqrt{n}}\,\mathscr{B}_n\left(\frac{i}{N}\right)\right) - \mathscr{B}_n(t)\right\} ds\right|^p dt\\
&= n^p \frac{1}{N} \sum_{i=0}^{N-1} \left|\mathscr{B}_n\left(\frac{i}{N}\right)\right|^p \int_0^1 \left|\int_0^1 \left\{\mathscr{B}_n\left(\frac{i+t}{N} - \frac{s}{\sqrt{n}}\,\mathscr{B}_n\left(\frac{i}{N}\right)\right) - \mathscr{B}_n\left(\frac{i+t}{N}\right)\right\} ds\right|^p dt\\
&=: n^p \frac{1}{N} \psi^{\circ}_{n,N}. \qquad (3.53)
\end{aligned}$$

Thus, statement (3.52)—which is, by the way, a statement concerning the Brownian bridge $\mathscr{B}_n$ only—can be rewritten in the following way

$$\lim_{n\to\infty} n^{p/4} \frac{\Psi^{\circ}_{n,N}}{\sum_{i=0}^{N-1} |\mathscr{B}_n(i/N)|^q} = \frac{1}{3^{p/2}}\, c_0^p(p) \quad a.s. \qquad (3.54)$$

We conclude this subsection by noting that statement (3.54) is the anticipated "discrete" version of statement (3.32). The proof of (3.54) is the topic of the next subsection.

3.3. *From the Brownian Bridges $\mathscr{B}_n$ to Other Brownian Bridges*

The main goal of this subsection is to enable ourselves to condition the right-hand side of (3.54) on $\mathscr{B}_n(i/N)$, $i = 0, 1, ..., N$. We start the realization of this idea by introducing the new processes

$$\begin{aligned}
\mathscr{B}_i^*(t) &:= \mathscr{B}^*_{i,n,N}(t)\\
&:= \sqrt{N}\left\{\mathscr{B}_n\left(\frac{i+t}{N}\right) - t\mathscr{B}_n\left(\frac{i+1}{N}\right) - (1-t)\,\mathscr{B}_n\left(\frac{i}{N}\right)\right\}, \qquad 0 \leqslant t \leqslant 1,
\end{aligned}$$

where $i = 0, 1, ..., N-1$. The processes $\mathscr{B}_i^*$, $i = 0, 1, ..., N-1$, are known (cf. Csörgő and Shi, 1998) to be independent Brownian bridges. Moreover, they are independent of all $\mathscr{B}_n(i/N)$, $i = 0, 1, ..., N$. Obviously, the integrand of $\Psi^{\circ}_{n,N}$ can be rewritten via the just introduced $\mathscr{B}_i^*$ as follows

$$\begin{aligned}
&\mathscr{B}_n\left(\frac{i+t}{N} - \frac{s}{\sqrt{n}}\,\mathscr{B}_n\left(\frac{i}{N}\right)\right) - \mathscr{B}_n\left(\frac{i+t}{N}\right)\\
&\quad = \frac{1}{\sqrt{N}}\left\{\mathscr{B}_i^*\left(t - s\frac{N}{\sqrt{n}}\,\mathscr{B}_n\left(\frac{i}{N}\right)\right) - \mathscr{B}_i^*(t)\right\}\\
&\qquad - s\frac{N}{\sqrt{n}}\,\mathscr{B}_n\left(\frac{i}{N}\right)\left\{\mathscr{B}_n\left(\frac{i+1}{N}\right) - \mathscr{B}_n\left(\frac{i}{N}\right)\right\}. \qquad (3.55)
\end{aligned}$$

The second summand (with the sign "$-$" in front of it) on the right-hand side of (3.55) is obviously asymptotically smaller than the first one. In order to demonstrate that the second summand is, indeed, so asymptotically small that it does not influence the result we want to achieve, we first obtain the following bound

$$\begin{aligned}
&\Bigg| \left| \int_0^1 \left\{ \mathscr{B}_n\left(\frac{i+t}{N} - \frac{s}{\sqrt{n}} \mathscr{B}_n\left(\frac{i}{N}\right)\right) - \mathscr{B}_n\left(\frac{i+t}{N}\right) \right\} ds \right|^p \\
&\qquad - \frac{1}{N^{p/2}} \left| \int_0^1 \left\{ \mathscr{B}_i^*\left(t - s\frac{N}{\sqrt{n}} \mathscr{B}_n\left(\frac{i}{N}\right)\right) - \mathscr{B}_i^*(t) \right\} ds \right|^p \Bigg| \\
&\quad \leqslant c \left| \frac{N}{\sqrt{n}} \mathscr{B}_n\left(\frac{i}{N}\right) \left\{ \mathscr{B}_n\left(\frac{i+1}{N}\right) - \mathscr{B}_n\left(\frac{i}{N}\right) \right\} \right|^p \\
&\qquad + c \left| \int_0^1 \left\{ \mathscr{B}_n\left(\frac{i+t}{N} - \frac{s}{\sqrt{n}} \mathscr{B}_n\left(\frac{i}{N}\right)\right) - \mathscr{B}_n\left(\frac{i+t}{N}\right) \right\} ds \right|^{p-1} \\
&\qquad \times \left| \frac{N}{\sqrt{n}} \mathscr{B}_n\left(\frac{i}{N}\right) \left\{ \mathscr{B}_n\left(\frac{i+1}{N}\right) - \mathscr{B}_n\left(\frac{i}{N}\right) \right\} \right|
\end{aligned} \tag{3.56}$$

that is an obvious consequence of representation (3.55) and inequality (3.39). We now estimate the right-hand side of (3.56) by using the law of the iterated logarithm (3.29) for $\|\mathscr{K}_n\|_\infty$ (recall that $\mathscr{B}_n = n^{-1/2}\mathscr{K}_n$ by definition) and W. Stute's result (3.37). In this way we derive from (3.56) the following representation

$$\begin{aligned}
&\left| \int_0^1 \left\{ \mathscr{B}_n\left(\frac{i+t}{N} - \frac{s}{\sqrt{n}} \mathscr{B}_n\left(\frac{i}{N}\right)\right) - \mathscr{B}_n\left(\frac{i+t}{N}\right) \right\} ds \right|^p \\
&\qquad = \frac{1}{N^{p/2}} \left| \int_0^1 \left\{ \mathscr{B}_i^*\left(t - s\frac{N}{\sqrt{n}} \mathscr{B}_n\left(\frac{i}{N}\right)\right) - \mathscr{B}_i^*(t) \right\} ds \right|^p + \delta_n^\circ(t),
\end{aligned} \tag{3.57}$$

where the remainder term δ_n° is such that

$$\|\delta_n^\circ\|_\infty \leqslant \left(\frac{N^{p/2}}{n^{p/2}} + \frac{N^{1/2}}{n^{(p+1)/4}} \right) \{ n^\epsilon N^\epsilon + o_{a.s.}(1) \} \tag{3.58}$$

for any fixed $\epsilon > 0$. Using representation (3.57) in the definition $\Psi_{n,N}^\circ$, we obtain the following representation

$$\begin{aligned}
\Psi_{n,N}^\circ &= \frac{1}{N^{p/2}} \sum_{i=0}^{N-1} \left| \mathscr{B}_n\left(\frac{i}{N}\right) \right|^p \int_0^1 \left| \int_0^1 \left\{ \mathscr{B}_i^*\left(t - s\frac{N}{\sqrt{n}} \mathscr{B}_n\left(\frac{i}{N}\right)\right) - \mathscr{B}_i^*(t) \right\} ds \right|^p dt \\
&\quad + \sum_{i=0}^{N-1} \left| \mathscr{B}_n\left(\frac{i}{N}\right) \right|^p \int_0^1 \delta_n^\circ(t)\, dt \\
&:= \Psi_{n,N}^{\circ\circ} + \delta_n^{\circ\circ}.
\end{aligned} \tag{3.59}$$

Because of (3.58), the remainder term $\delta_n^{\circ\circ}$ is obviously such that

$$|\delta_n^{\circ\circ}| \leqslant cN \left(\frac{N^{p/2}}{n^{p/2}} + \frac{N^{1/2}}{n^{(p+1)/4}} \right) \{ n^{\epsilon} N^{\epsilon} + o_{a.s.}(1) \} \tag{3.60}$$

for any fixed $\epsilon > 0$. Using the just obtained bound (3.60), we arrive at the statement

$$\lim_{n \to \infty} n^{p/4} \frac{|\delta_n^{\circ\circ}|}{\sum_{i=0}^{N-1} |\mathscr{B}_n(i/N)|^q} = 0 \quad a.s., \tag{3.61}$$

provided that we chose $\epsilon > 0$ sufficiently small, which we do. Statement (3.61) and representation (3.59) imply that statement (3.54) follows from the following one

$$\lim_{n \to \infty} n^{p/4} \frac{\Psi_{n,N}^{\circ\circ}}{\sum_{i=0}^{N-1} |\mathscr{B}_n(i/N)|^q} = \frac{1}{3^{p/2}} c_0^p(p) \quad a.s. \tag{3.62}$$

In next Subsection 3.4 we shall prove (3.62).

3.4. *From the Brownian Bridges $\mathscr{B}_n^*$ to Statement* (2.12)

By the Borel–Cantelli lemma, statement (3.62) holds true if, for every $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} \mathbf{P} \left\{ \left| n^{p/4} \frac{\Psi_{n,N}^{\circ\circ}}{\sum_{i=0}^{N-1} |\mathscr{B}_n(i/N)|^q} - \frac{1}{3^{p/2}} c_0^p(p) \right| \geqslant \varepsilon \right\} < \infty. \tag{3.63}$$

To proceed with the proof of (3.63), we first simplify notations by letting

$$y_i := \mathscr{B}_n(i/N) \tag{3.64}$$

for all $i = 0, 1, ..., N$. Furthermore, let $\mathbf{P}^*$ denote the conditional probability with respect to all $y_0, y_1, ..., y_N$. Then the probability $\mathbf{P}\{ \cdots \}$ of (3.63) can be expressed in the following way

$$\begin{aligned} &\mathbf{P} \left\{ \left| n^{p/4} \frac{\Psi_{n,N}^{\circ\circ}}{\sum_{i=0}^{N-1} |\mathscr{B}_n(i/N)|^q} - \frac{1}{3^{p/2}} c_0^p(p) \right| \geqslant \varepsilon \right\} \\ &\quad = \mathbf{E} \left(\mathbf{P}^* \left\{ \left| \sum_{i=0}^{N-1} |y_i|^q \, Y_i \right| \geqslant \sum_{i=0}^{N-1} |y_i|^q \, \varepsilon \right\} \right), \end{aligned} \tag{3.65}$$

where

$$Y_i := \left\{ \frac{\sqrt{n}}{N\,|y_i|} \right\}^{p/2} \int_0^1 \left| \int_0^1 \left\{ \mathscr{B}_i^* \left(t - s \frac{N}{\sqrt{n}} y_i \right) - \mathscr{B}_i^*(t) \right\} ds \right|^p dt - \frac{1}{3^{p/2}} c_0^p(p).$$

Consequently, statement (3.63) is equivalent to the following one

$$\sum_{n=1}^{\infty} \mathbf{E}\left(\mathbf{P}^* \left\{\left|\sum_{i=0}^{N-1} |y_i|^q\, Y_i\right| \geqslant \sum_{i=0}^{N-1} |y_i|^q\, \varepsilon\right\}\right) < \infty. \tag{3.66}$$

The conditional probability $\mathbf{P}^*\{\cdots\}$ of (3.66) does not exceed

$$\mathbf{P}^* \left\{\left|\sum_{i=0}^{N-1} |y_i|^q\, (Y_i - \mu_i)\right| \geqslant \sum_{i=0}^{N-1} |y_i|^q\, (\varepsilon - |\mu_i|)\right\} \tag{3.67}$$

where we have denoted

$$\mu_i := \mathbf{E}\, Y_i.$$

Furthermore, since

$$\sum_{n=1}^{\infty} \mathbf{P}\{\max_{i=0,1,\ldots,N-1} |y_i| \geqslant \log n\}$$
$$\leqslant \sum_{n=1}^{\infty} \mathbf{P}\{\|\mathscr{B}_n\| \geqslant \log n\} = \sum_{n=1}^{\infty} \exp\{-2(\log n)^2\} < \infty, \tag{3.68}$$

statement (3.66) holds true if the following one

$$\sum_{n=1}^{\infty} \mathbf{E}\left(\mathbf{I}\{\max_{i=0,1,\ldots,N-1} |y_i| \leqslant \log n\}\ \mathbf{P}^* \left\{\left|\sum_{i=0}^{N-1} |y_i|^q\, (Y_i - \mu_i)\right|\right.\right.$$
$$\left.\left.\geqslant \sum_{i=0}^{N-1} |y_i|^q\, (\varepsilon - |\mu_i|)\right\}\right) < \infty \tag{3.69}$$

does. In order to prove (3.69) we assume from now on that

$$\max_{i=0,1,\ldots,N-1} |y_i| \leqslant \log n. \tag{3.70}$$

We shall prove in Subsection 3.5 that the following bound

$$|\mu_i| \leqslant c \left\{\frac{N}{\sqrt{n}}\, |y_i|\right\}^{1/2} \tag{3.71}$$

holds true under assumption (3.71). Taking now (3.71) for granted, we choose $\epsilon_0 > 0$ in the definition of N sufficiently small and immediately derive from (3.71) that bound

$$|\mu_i| \leqslant \frac{\epsilon}{2} \tag{3.72}$$

holds true for all sufficiently large $n \in \mathbf{N}$, say, for all $n \geqslant n_0$. Obviously though, the assumption $n \geqslant n_0$ does not restrict the generality, since we can always delete a finite number of summands from (3.69) without affecting the validity of our considerations. Using (3.72) and Chebyshev's inequality, we obviously see that statement (3.69) follows if the quantities

$$\left\{ \sum_{i=0}^{N-1} |y_i|^q \right\}^{-v} \mathbf{E} \left| \sum_{i=0}^{N-1} |y_i|^q \, (Y_i - \mu_i) \right|^v \tag{3.73}$$

are summable over all $n \in \mathbf{N}$, for some fixed $v \geqslant 0$. Considering only $v \geqslant 3$ and, consequently, being able to apply the well-known von Bahr inequality (cf., for example, Petrov, 1975), we estimate the quantity of (3.73) by the following one

$$\left\{ \sum_{i=0}^{N-1} |y_i|^q \right\}^{-v} \left\{ \sum_{i=0}^{N-1} |y_i|^{2q} \, \mathbf{E}(Y_i - \mu_i)^2 \right\}^v \left\{ \mathbf{E}\, |G|^v + \mathcal{O}\left(\frac{1}{\sqrt{n}} \right) \right\}, \tag{3.74}$$

where G is a Gaussian random variable with mean 0 and variance 1. Since $|y_i|^q \leqslant (\log n)^q$ by (3.70), the quantity of (3.74) does not exceed

$$c\{\log n\}^{qv} \left\{ \sum_{i=0}^{N-1} |y_i|^q \right\}^{-v} \left\{ \sum_{i=0}^{N-1} |y_i|^q \, \mathbf{E}(Y_i - \mu_i)^2 \right\}^v \tag{3.75}$$

for a finite constant $c > 0$ that depends only on v. Obviously, the expectation $\mathbf{E}(Y_i - \mu_i)^2$ does not exceed $\mathbf{E} Y_i^2$ which, in turn, does not exceed

$$n^{1/8} \, |\mathbf{E} Y_i| + \frac{1}{n^{1/8}} \mathbf{E}\, |Y_i|^3. \tag{3.76}$$

The random variable $|Y_i|^3$ does not exceed $Z_i + c^*$ almost surely, where Z_i stands for the random variable Y_i with p replaced by $3p$, and c^* is a finite constant depending only on p. Consequently, we obtain from bound (3.71) that

$$\mathbf{E}\, |Y_i|^3 \leqslant \mathbf{E} Z_i + c^* \leqslant c \left\{ \frac{N}{\sqrt{n}} \, |y_i| \right\}^{1/2} + c^*. \tag{3.77}$$

Using bound (3.71) for estimating $|\mathbf{E} Y_i|$ and bound (3.77) for estimating $\mathbf{E}\, |Y_i|^3$, we derive via (3.76) the first of the following two inequalities

$$\mathbf{E}(Y_i - \mu_i)^2 \leqslant c n^{1/8} \left\{ \frac{N}{\sqrt{n}} \, |y_i| \right\}^{1/2} + c^* \frac{1}{n^{1/8}} \leqslant c n^{1/8} \left\{ \frac{N}{\sqrt{n}} \log n \right\}^{1/2} + c^* \frac{1}{n^{1/8}}, \tag{3.78}$$

whereas the second one is a consequence of (3.70). Choosing $\epsilon_0 > 0$ (in the definition of N) sufficiently small and the parameter v in (3.75) sufficiently large, we obviously achieve the summability of (3.75) over $n \in \mathbf{N}$. This completes the proof of statement (3.63), and thus of (2.12) as well. Let us recall, however, that we still need to prove the validity of bound (3.71); this is the topic of our next subsection.

3.5. *Proof of Bound* (3.71)

With the following notation

$$h := \frac{N}{\sqrt{n}} \, |y_i|$$

the proof of bound (3.71) becomes much more elegant. Thus, our goal is to show that the bound

$$|\mu_i| \leqslant c h^{1/2} \tag{3.79}$$

holds true. In fact, we shall demonstrate the validity of bound under the assumption $h \leqslant 1$ which, however, is always fulfilled for all sufficiently large n by the comment just below (3.71) and thus does not restrict the generality of our considerations. We start off the proof of (3.79) by noting that

$$\mu_i = \frac{1}{h^{p/2}} \int_0^1 \mathbf{E} \left| \int_0^1 \{\mathscr{B}_i^*(t \pm sh) - \mathscr{B}_i^*(t)\} \, ds \right|^p dt - \frac{1}{3^{p/2}} c_0^p(p), \tag{3.80}$$

where "$\pm$" means "$+$" if $y_i \leqslant 0$, and "$-$" if $y_i > 0$. In order to fully appreciate and use a somewhat implicitly present in (3.80) independence structure, we now replace the Brownian bridge $\mathscr{B}_i^*$ in (3.80) by the corresponding Wiener process $\mathscr{W}_i^*$ defined via the equality $\mathscr{B}_i^*(t) = \mathscr{W}_i^*(t) - t\mathscr{W}_i^*(1)$. Obviously,

$$\mathscr{B}_i^*(t \pm sh) - \mathscr{B}_i^*(t) = \{\mathscr{W}_i^*(t \pm sh) - \mathscr{W}_i^*(t)\} \mp sh\mathscr{W}_i^*(1). \tag{3.81}$$

The last summand $sh\mathscr{W}_i^*(1)$ on the right-hand side of (3.81) is obviously asymptotically smaller (when $h \to 0$) than the first one $\mathscr{W}_i^*(t \pm sh) - \mathscr{W}_i^*(t)$. This fact leads us to the idea of expressing μ_i in the following way

$$\mu_i = \mu_i^\diamond + \delta_{i,n}^\diamond, \tag{3.82}$$

where

$$\mu_i^{\diamond} := \frac{1}{h^{p/2}} \int_0^1 \mathbf{E} \left| \int_0^1 \{ \mathscr{W}_i^*(t \pm sh) - \mathscr{W}_i^*(t) \} \, ds \right|^p dt - \frac{1}{3^{p/2}} c_0^p(p). \tag{3.83}$$

The remainder term $\delta_{i,n}^{\diamond}$, which obviously equals to $\mu_i - \mu_i^{\diamond}$ by (3.82), can be estimated using bound (3.39) as follows

$$|\delta_{i,n}^{\diamond}| \leqslant c \frac{1}{h^{p/2}} \int_0^1 \mathbf{E} \left\{ |h \mathscr{W}_i^*(1)|^p + \left| \int_0^1 \{ \mathscr{W}_i^*(t \pm sh) - \mathscr{W}_i^*(t) \} \, ds \right|^{p-1} |h \mathscr{W}_i^*(1)| \right\} dt. \tag{3.84}$$

if $p = 1$, then, obviously, $|\delta_{i,n}^{\diamond}| \leqslant c \sqrt{h}$. If $p < 1$, then

$$|\delta_{i,n}^{\diamond}| \leqslant c h^{p/2} + c \frac{1}{h^{(p-2)/2}} \int_0^1 \left\{ \mathbf{E} \left| \int_0^1 \{ \mathscr{W}_i^*(t \pm sh) - \mathscr{W}_i^*(t) \} \, ds \right|^{r(p-1)} \right\}^{1/r} dt \tag{3.85}$$

for any $r > 1$. Since $p > 1$ by assumption, we can always choose $r > 1$ so large that $r(p-1) > 1$. With such $r > 1$ we have

$$\mathbf{E} \left| \int_0^1 \{ \mathscr{W}_i^*(t \pm sh) - \mathscr{W}_i^*(t) \} \, ds \right|^{r(p-1)} \leqslant \int_0^1 \mathbf{E} \, |\mathscr{W}_i^*(t \pm sh) - \mathscr{W}_i^*(t)|^{r(p-1)} \, ds. \tag{3.86}$$

The difference $|\mathscr{W}_i^*(t \pm sh) - \mathscr{W}_i^*(t)|$ has the same distribution as $\sqrt{sh}\,|G|$, where G is a Gaussian random variable with mean 0 and variance 1. Consequently, the quantity on the right-hand side of (3.86) does not exceed $ch^{r(p-1)/2}$, and thus we obtain the bound

$$\mathbf{E} \left| \int_0^1 \{ \mathscr{W}_i^*(t \pm sh) - \mathscr{W}_i^*(t) \} \, ds \right|^{r(p-1)} \leqslant c h^{r(p-1)/2}. \tag{3.87}$$

When applied on the right-hand side of (3.85), bound (3.87) implies the first of the following two bounds

$$|\delta_{i,n}^{\diamond}| \leqslant c h^{p/2} + c h^{1/2} \leqslant c h^{1/2}, \tag{3.88}$$

whereas the second bound of (3.88) follows from our assumption $h \leqslant 1$ and the fact that $p \geqslant 1$.

We are now to demonstrate that

$$\mu_i^{\diamond} = 0. \tag{3.89}$$

We start the proof of (3.89) by replacing the integration variable t by ht in the definition of $\mu_i^{\diamond}$. In this way we obtain that

$$\mu_i^{\diamond} = h \int_0^{1/h} \mathbf{E} \left| \int_0^1 \{\mathscr{W}_i^{**}(t \pm s) - \mathscr{W}_i^{**}(t)\} \, ds \right|^p dt - \frac{1}{3^{p/2}} c_0^p(p), \tag{3.90}$$

where

$$\mathscr{W}_i^{**}(u) := \frac{1}{\sqrt{h}} \mathscr{W}_i^{*}(hu). \tag{3.91}$$

To make notations even more simpler, from now on we use W instead of $\mathscr{W}_i^{**}$. Our further task is to show that

$$\mu := \mathbf{E} \left| \int_0^1 \{W(t \pm s) - W(t)\} \, ds \right|^p = \frac{1}{3^{p/2}} c_0^p(p) \tag{3.92}$$

which will complete the proof of (3.89). As the first step towards the equality of (3.92), we demonstrate that

$$\mu = \lim_{M \to \infty} \mu_M(p), \tag{3.93}$$

where

$$\mu_M(p) := \mathbf{E} \left| \frac{1}{M} \sum_{m=0}^{M-1} \left\{ W\left(t \pm \frac{m}{M}\right) - W(t) \right\} \right|^p .$$

For this reason, we rewrite μ in the following form

$$\begin{aligned}
\mu &= \mathbf{E} \left| \sum_{m=0}^{M-1} \int_{m/M}^{(m+1)/M} \{W(t \pm s) - W(t)\} \, ds \right|^p \\
&= \mathbf{E} \left| \frac{1}{M} \sum_{m=0}^{M-1} \int_0^1 \left\{ W\left(t \pm \frac{m+s}{M}\right) - W\left(t \pm \frac{m}{M}\right) \right\} ds \right. \\
&\quad \left. + \frac{1}{M} \sum_{m=0}^{M-1} \left\{ W\left(t \pm \frac{m}{M}\right) - W(t) \right\} \right|^p .
\end{aligned} \tag{3.94}$$

It is now clear that, due to (3.39),

$$\begin{aligned}
&|\mu-\mu_M(p)|\\
&\quad\leqslant c\mathbf{E}\left|\frac{1}{M}\sum_{m=0}^{M-1}\int_0^1\left\{W\left(t\pm\frac{m+s}{M}\right)-W\left(t\pm\frac{m}{M}\right)\right\}ds\right|^p\\
&\quad+c\sqrt{\mu_M(2p-2)}\left\{\mathbf{E}\left|\frac{1}{M}\sum_{m=0}^{M-1}\int_0^1\left\{W\left(t\pm\frac{m+s}{M}\right)-W\left(t\pm\frac{m}{M}\right)\right\}ds\right|^2\right\}^{1/2}.
\end{aligned}\tag{3.95}$$

By the Hölder inequality and the fact that the random variable $W(t+\tau)-W(t)$ has the same distribution as a Gaussian random variable with mean 0 and variance τ, we get that

$$\mathbf{E}\left|\frac{1}{M}\sum_{m=0}^{M-1}\int_0^1\left\{W\left(t\pm\frac{m+s}{M}\right)-W\left(t\pm\frac{m}{M}\right)\right\}ds\right|^r\leqslant c\frac{1}{M^{r/2}}\tag{3.96}$$

for any $r\geqslant 1$ and a finite constant c depending only on r. Thus, using bound (3.96) first with $r=p$ and then with $r=2$ on the right-hand side of (3.95), we obtain that

$$|\mu-\mu_M(p)|\leqslant c\frac{1}{M^{p/2}}+c\sqrt{\mu_M(2p-2)}\frac{1}{M^{1/2}}.\tag{3.97}$$

We are now to exactly evaluate $\mu_M(r)$ for any $r\geqslant 0$. Obviously,

$$\begin{aligned}
\mu_M(r)&=\mathbf{E}\left|\frac{1}{M}\sum_{m=0}^{M-1}\sum_{j=0}^{m-1}\left\{W\left(t\pm\frac{j+1}{M}\right)-W\left(t\pm\frac{j}{M}\right)\right\}\right|^r\\
&=\mathbf{E}\left|\frac{1}{M}\sum_{m=0}^{M-1}(M-j)\left\{W\left(t\pm\frac{j+1}{M}\right)-W\left(t\pm\frac{j}{M}\right)\right\}\right|^r\\
&=\mathbf{E}\left|\frac{1}{M^{3/2}}\sum_{m=0}^{M-1}(M-j)\,G_j\right|^r,
\end{aligned}\tag{3.98}$$

where we have denoted

$$G_j:=\sqrt{M}\left\{W\left(t\pm\frac{j+1}{M}\right)-W\left(t\pm\frac{j}{M}\right)\right\}.$$

Note that $G_0, \ldots, G_{M-1}$ are independent Gaussian random variables with means 0 and variances 1. The characteristic function of the random variable

$$\xi := M^{-3/2} \sum_{m=0}^{M-1} (M-j)\, G_j$$

coincides with that of the random variable

$$\eta := \left\{ M^{-3} \sum_{m=0}^{M} m^2 \right\}^{1/2} G,$$

where G is a Gaussian random variable with mean 0 and variance 1. Thus, the random variables ξ and η have identical distribution functions. This fact, in turn, implies that

$$\begin{aligned} \mathbf{E} \left| \frac{1}{M^{3/2}} \sum_{m=0}^{M-1} (M-j)\, G_j \right|^r &= \left\{ \frac{1}{M^3} \sum_{m=0}^{M} m^2 \right\}^{r/2} \mathbf{E}\, |G|^r \\ &= \left\{ \frac{1}{M^3} \sum_{m=0}^{M} m^2 \right\}^{r/2} c_0^r(r). \end{aligned} \tag{3.99}$$

Consequently, (3.99) and (3.98) imply the equality

$$\mu_M(r) = \left\{ \frac{1}{M^3} \sum_{m=0}^{M} m^2 \right\}^{r/2} c_0^r(r) \tag{3.100}$$

for any $r \geqslant 0$. Applying now equality (3.100) with $r = 2p-2$ on the right-hand side of (3.97), we get the bound

$$|\mu - \mu_M(p)| \leqslant c\, \frac{1}{M^{p/2}} + c\, \frac{1}{M^{1/2}}$$

which, in turn, immediately implies statement (3.93). On the other hand, equality (3.100) with $r = p$ obviously shows that the right hand side of (3.93) equals to $3^{-p/2} c_0^p(p)$ which, in turn, implies that the equality of (3.92) holds true. Consequently, statement (3.89) holds true as well. This also completes the proof of bound (3.71).

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